This chapter deals with Riemann surfaces, coverings, and hypergeometric functions. It first considers the genus and Euler number of a Riemann surface before discussing Möbius transformations and notes that an automorphism of a Riemann surface is a biholomorphic map of the Riemann surface onto itself. It then describes a Riemannian metric and the Gauss-Bonnet theorem, which can be interpreted as a relation between the Gaussian curvature of a compact Riemann surface $X$ and its Euler characteristic. It also examines the behavior of the Euler number under finite covering, along with finite subgroups of the group of fractional linear transformations $\text{PSL}(2, \mathbb{C})$. Finally, it presents some basic facts about the classical Gauss hypergeometric functions of one complex variable, triangle groups acting discontinuously on one of the simply connected Riemann surfaces, and the hypergeometric monodromy group.

**THE VIRTUES OF CRACKED REASONING**

Mark Wilson

Drawing upon the rich experience gathered within applied mathematics, various ‘facade patterns’ are examined that frequently develop when an originating usage enlarges its descriptive scope through patch-to-patch prolongation. The completed results can generate a global
structure that is syntactically inconsistent as a whole, yet avoids logical ruination through simple restrictions upon data exportation from one patch to another (a ‘Riemann surface’ represents a standard mathematical prototype of the phenomenon). It is argued that not only do such facades often represent the natural end products of ordinary linguistic development, they often provide particularly effective forms of linguistic engineering. Philosophical puzzles sometimes arise when these alternative patterns of semantic design get mistaken for classical models, as the troubled history of ‘force’ effectively illustrates. The fact that we can rarely determine whether an initial collection of descriptive vocabulary is destined to develop into a facade rather than implementing a simpler pattern of word/world alignment provides a convenient indication of the degree to which a classical picture of conceptual grasp exaggerates our capacity to augur the fate of our descriptive words over time.

Multivalued harmonic morphisms
Paul Baird and John C. Wood

in Harmonic Morphisms Between Riemannian Manifolds

Multivalued analytic functions are considered to be functions on a Riemann surface, which can be defined either by cutting and pasting or by a graph construction. Multivalued maps in general are considered, and it is shown how the concept of multivalued analytic functions generalizes to harmonic morphisms using the graph construction. An alternative treatment for space forms is presented, and some specific examples are given. The chapter concludes with a discussion of the behaviour on the branching set of the projection map for a multivalued harmonic morphism on a three-dimensional space form.

Meromorphic functions and the Main Theorem for compact Riemann surfaces
Simon Donaldson

in Riemann Surfaces

Meromorphic functions and the Main Theorem for compact Riemann surfaces
Simon Donaldson

Page 2 of 9
This chapter explains a strategy for proving the fundamental structural results about Riemann surfaces. It first considers the consequence of the Main Theorem for compact Riemann surfaces. It then analyzes the Riemann-Roch formula, the fundamental tool in the theory of compact Riemann surfaces.

Holomorphic harmonic morphisms
Paul Baird and John C. Wood

in Harmonic Morphisms Between Riemannian Manifolds

Published in print: 2003 Published Online: September 2007
Item type: chapter

The first section of this chapter investigates the general question of when holomorphic maps between almost Hermitian manifolds are harmonic maps or harmonic morphisms. In particular, a holomorphic map from a Kaehler manifold to a Riemann surface is always a harmonic morphism. It is shown how harmonic morphisms into a Riemann surface can sometimes be combined to give new ones. The construction of harmonic morphisms from domains of Euclidean and related spaces is discussed, which are holomorphic with respect to a Hermitian structure, thus finding interesting globally defined complex-valued harmonic morphisms on a Euclidean space of arbitrary dimension.

TOPOLOGICAL STRINGS
Marcos Mariño

in Chern-Simons Theory, Matrix Models, and Topological Strings

Published in print: 2005 Published Online: September 2007
Item type: chapter

Type-A and type-B topological sigma models are two topological field theories in two dimensions. Although they contain a lot of information in genus 0, they turn out to be trivial for \( g > 1 \). This is essentially due to the fact that, in order to define these theories, it is necessary to consider a fixed metric in the Riemann surface. In order to obtain a non-trivial theory in higher genus the degrees of freedom of the two-dimensional metric must be introduced. This means that the topological field theories must be coupled to two-dimensional gravity. The coupling to gravity is done by using the fact that the structure of the twisted theory is tantalizingly close to that of the bosonic string. Topological
sigma models may be defined not only on closed Riemann surfaces and closed topological strings, but also on the open case.

**Contrasts in Riemann surface theory**

Simon Donaldson

in Riemann Surfaces

Section 4.2.3 showed how to associate a compact connected Riemann surface to any irreducible polynomial $P(z,w)$ in two variables. This chapter first establishes the converse — any compact connected Riemann surface arises from a polynomial $P$. It also explains how the whole theory can be cast in an algebraic form, involving fields of transcendence degree 1. The chapter then analyzes hyperbolic surfaces, covering models of the hyperbolic plane; self-isometries; geodesics; the Gauss-Bonnet Theorem; right-angled hexagons; and closed geodesics.

**Riemann Surfaces**

Simon Donaldson

The theory of Riemann surfaces occupies a very special place in mathematics. It is a culmination of much of traditional calculus, making surprising connections with geometry and arithmetic. It is an extremely useful part of mathematics, knowledge of which is needed by specialists in many other fields. It provides a model for a large number of more recent developments in areas including manifold topology, global analysis, algebraic geometry, Riemannian geometry, and diverse topics in mathematical physics. This text on Riemann surface theory proves the fundamental analytical results on the existence of meromorphic functions and the Uniformisation Theorem. The approach taken emphasises PDE methods, applicable more generally in global analysis. The connection with geometric topology, and in particular the role of the mapping class group, is also explained. To this end, some more sophisticated topics have been included, compared with traditional texts at this level. While the treatment is novel, the roots of the subject in traditional calculus and complex analysis are kept well in mind. Part I sets up the interplay between complex analysis and topology, with the latter treated
informally. Part II works as a rapid first course in Riemann surface theory, including elliptic curves. The core of the book is contained in Part III, where the fundamental analytical results are proved.

Facades
Christopher Pincock

in Mathematics and Scientific Representation

Mark Wilson’s book Wandering Significance: An Essay on Conceptual Behavior is considered in this chapter. Wilson argues that common views on mathematical and scientific concepts distort our understanding of scientific knowledge and representation. Pincock substantially agrees with Wilson and describes how the positions from earlier chapters can be adapted to fit with Wilson’s worries. At the heart of Wilson’s worries is a view of representations as made up of a collection of locally effective patches. Pincock explains how mathematics generates these patches and helps to link them together. The example of the development of the concept of Riemann surfaces is used to further clarify this picture of scientific representation and to vindicate a form of patient scientific realism.

Basic definitions
Simon Donaldson

in Riemann Surfaces

This chapter presents definitions of Riemann surfaces and holomorphic maps, and provides examples, including algebraic curves and quotients.

Functions of a complex variable
Chun Wa Wong

in Introduction to Mathematical Physics: Methods & Concepts
Functions of a complex variable are shown to be more complete and rigid than functions of a real variable. Analytic functions with well defined derivatives satisfy two Cauch–Riemann conditions. Multivalued functions can be made single-valued on a multi-sheet Riemann surface. The values of an analytic function in a region of the complex plane are completely defined by the knowledge of their values on a closed boundary of the region. Important properties and techniques of complex analysis are described. These include Taylor and Laurent expansions, contour integration and residue calculus, Green's functions, Laplace transforms and Bromwich integrals, dispersion relations and asymptotic expansions. Analytic functions are defined by their properties at the locations called singularities (poles and branch cuts) where they cease to be analytic. This feature makes analytic functions of particular interest in the construction of physical theories.

Elliptic functions and integrals
Simon Donaldson

in Riemann Surfaces

This chapter examines Riemann surfaces of genus 1. The constructions give an important model for the more general theory to be developed in Part III. The constructions also involve classical topics in mathematics, which relate the abstractions of Riemann surface theory to their origin in concrete calculus problems.

Calculus on surfaces
Simon Donaldson

in Riemann Surfaces

This chapter develops the theory of differential forms on smooth surfaces and Riemann surfaces. The analysis begins with smooth surfaces, covering cotangent spaces and 1-forms; 2-forms and integration. It then turns to de Rham cohomology, which includes cohomology with compact support, and Poincaré duality. The final section discusses calculus on
Riemann surfaces, covering decomposition of the 1-forms; the Laplace operator and harmonic functions; and the Dirichlet norm.

**Entropy for hyperbolic Riemann surface laminations I**

Tien-Cuong Dinh, Viet-Anh Nguyen, and Nessim Sibony

Araceli Bonifant, Mikhail Lyubich, and Scott Sutherland (eds)

in Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday

Published in print: 2014 Published Online: October 2017


This chapter introduces a notion of entropy for possibly singular hyperbolic laminations by Riemann surfaces. It also studies the transverse regularity of the Poincaré metric and the finiteness of the entropy. The chapter first focuses on compact laminations, which are transversally smooth, before turning to the case of singular foliations, showing how the Poincaré metric on leaves is transversally Hölder continuous. In addition, the chapter considers the problem in the proof that the entropy is finite for singular foliations is quite delicate and requires a careful analysis of the dynamics around the singularities. Finally, the chapter discusses a notion of metric entropy for harmonic probability measures and gives some open questions.

**The Uniformisation Theorem**

Simon Donaldson

in Riemann Surfaces

Published in print: 2011 Published Online: December 2013


This chapter proves the following theorem. Theorem 12: Let X be a connected, simply connected, non-compact Riemann surface. Then X is equivalent to either C or the upper half-plane H. The proof presented here follows the same general pattern as one already given to classify compact simply connected Riemann surfaces, but the non-compactness will require some extra steps.
Holomorphic functions
Simon Donaldson

in Riemann Surfaces

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Item type: chapter

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acprof:oso/9780198526391.003.0001

This chapter reviews some examples of holomorphic functions in complex analysis. It emphasizes the idea of ‘analytic continuation’, which is a fundamental motivation for Riemann surface theory.

Entropy for hyperbolic Riemann surface laminations II
Tien-Cuong Dinh, Viet-Anh Nguyen, and Nessim Sibony

Araceli Bonifant, Mikhail Lyubich, and Scott Sutherland (eds)
in Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday

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This chapter studies Riemann surface foliations with tame singular points. It shows that the hyperbolic entropy of a Brody hyperbolic foliation by Riemann surfaces with linearizable isolated singularities on a compact complex surface is finite. The chapter then proves the finiteness of the entropy in the local setting near a singular point in any dimension, using a division of a neighborhood of a singular point into adapted cells. Next, the chapter estimates the modulus of continuity for the Poincaré metric along the leaves of the foliation, using notion of conformally (R,δ‎)-close maps. The estimate holds for foliations on manifolds of higher dimension.

Proof of the Main Theorem
Simon Donaldson

in Riemann Surfaces

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acprof:oso/9780198526391.003.0009

This chapter provides proof of the main analytical result, Theorem 5, for compact Riemann surfaces.

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This chapter explains the correspondence between local systems on a punctured Riemann surface with the structure group being a real reductive Lie group G, and parabolic G-Higgs bundles. The chapter describes the objects involved in this correspondence, taking some time to motivate them by recalling the definitions of G-Higgs bundles without parabolic structure and of parabolic vector bundles. Finally, it explains the relevant polystability condition and the correspondence between local systems and Higgs bundles.

Brauer Group of Moduli of Higgs Bundles and Connections

David Baraglia, Indranil Biswas, and Laura P. Schaposnik

Given a compact Riemann surface X and a semi-simple affine algebraic group G defined over C, there are moduli spaces of Higgs bundles and of connections associated to (X, G). The chapter computes the Brauer group of the smooth locus of these varieties.